Inverse trig functions

Although the basic sine function isn't invertible (i.e., it doesn't have an inverse), we can restrict its domain in such a way that the restricted version of the sine function is invertible. To do that, we have to restrict the domain to a set S of angles such that for every x in the interval $[-1,1]$, there is one and only one angle $\theta \in S$ with $\sin\theta = x$.

As you may recall, the sine function is positive in the first and second quadrants, and it takes on every value x in the interval $(0,1)$ exactly once in each of those two quadrants. Also, the sine function is negative in the third and fourth quadrants, and it takes on every value x in the interval $(-1,0)$ exactly once in each of those two quadrants. In addition,

$$\sin\left(\frac{\pi}{2}\right) = 1 \qquad \sin(0) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1$$

We would like range of the inverse sine function to be a "connected" set (i.e., a single interval). Now recall that every angle α in the interval $[0,2\pi)$ which is in the fourth quadrant is in the interval $(3\pi/2,2\pi)$, and notice that the angle $\theta = \alpha - 2\pi$ is in the interval $(-\pi/2,0)$ and is coterminal with α, so

$$\sin\theta = \sin\alpha$$

Also, the angle of measure $3\pi/2$ is coterminal with the angle of measure $-\pi/2$, so

$$\sin\left(-\frac{\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right)$$

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Thus the natural (and standard) way to restrict the domain of the sine function (so that the restricted version has an inverse and takes on every value x in the interval $[-1,1]$ exactly once) is to restrict it to the interval $[-\pi/2,\pi/2]$.

A similar thing happens with the cosine function. Recall that the cosine function is positive in the first and fourth quadrants, and that it takes on every value x in the interval $(0,1)$ exactly once in each of those two quadrants. Also, the cosine function is negative in the second and third quadrants, and it takes on every value x in the interval $(-1,0)$ in each of those two quadrants. Besides that,

$$\cos(0) = 1 \qquad \cos\left(\frac{\pi}{2}\right) = 0 \qquad \cos(\pi) = -1$$

Again, we would like the range of the inverse cosine function to be an interval. Thus the natural (and standard) way to restrict the domain of the cosine function (so that the restricted version has an inverse and takes on every value x in the interval $[-1,1]$ exactly once) is to restrict it to the interval $[0,\pi]$.

Notice that the domains of the (standard) restricted sine and cosine functions are different. The domain of the restricted sine function is the set of all angles θ in the interval $[-\pi/2,\pi/2]$, and the domain of the restricted cosine function is the set of all angles θ in the interval $[0,\pi]$.

The notation that's used for the inverse of the (restricted) sine function is $\sin^{-1}(x)$, and the notation that's used for the inverse of the (restricted) cosine function is $\cos^{-1}(x)$. The superscript -1 means “inverse function” (of

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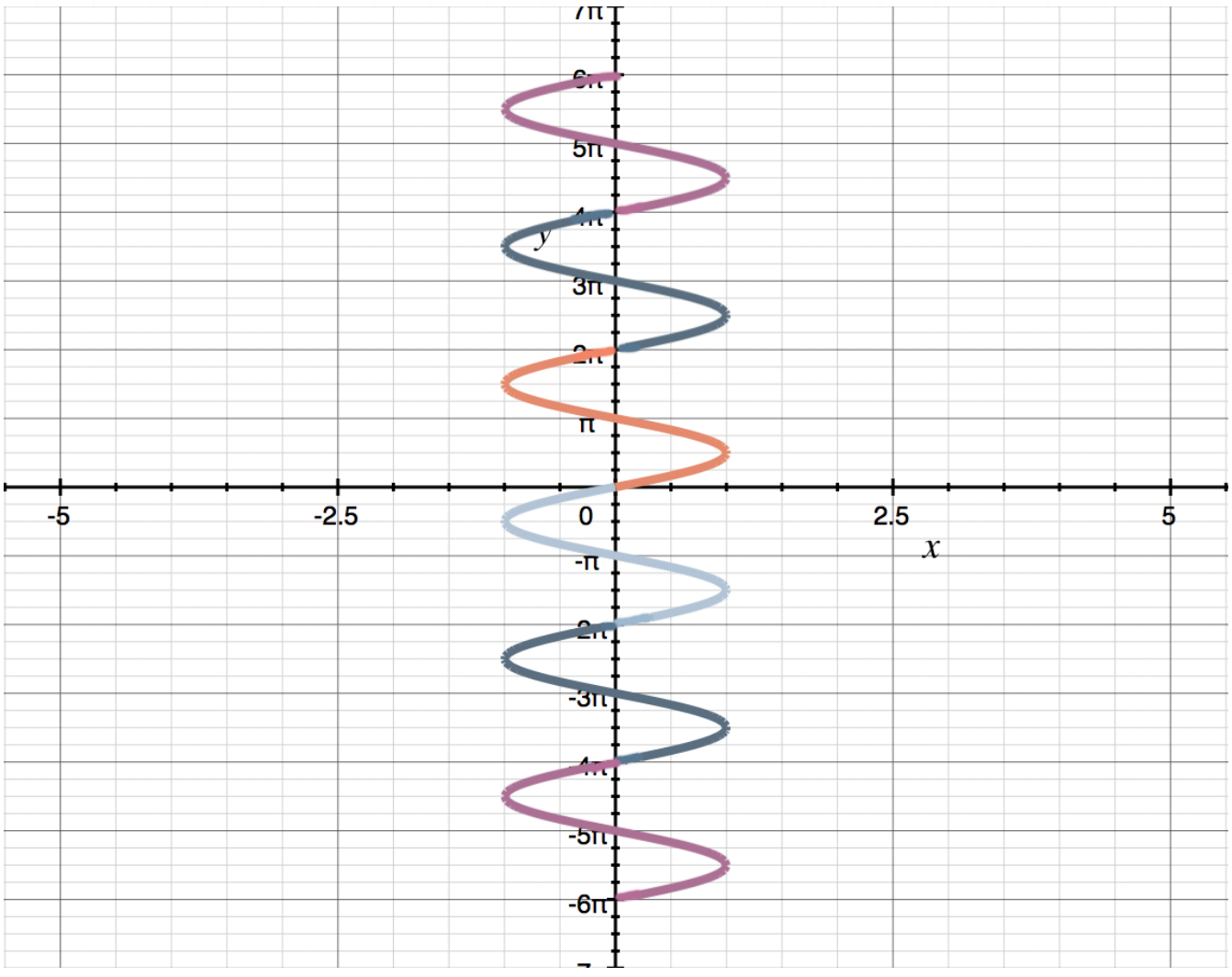
the sine or cosine function), not “reciprocal of the value” (of the sine or cosine function).

The domain of the inverse of a function is equal to the range of that function, and vice versa. Thus the domain of the function $\sin^{-1}(x)$ is $[-1,1]$, and its range is $[-\pi/2,\pi/2]$. Similarly, the domain of the function $\cos^{-1}(x)$ is $[-1,1]$, and its range is $[0,\pi]$.

The graph of the inverse sine function is identical to the part of the graph of the inverse sine relation that corresponds to angles in the interval $[-\pi/2,\pi/2]$. To see this, we'll first display the graph of the inverse sine relation.



Next, we'll use that to graph the inverse sine function, which consists of the first half of the part of the graph of the inverse sine relation that's in

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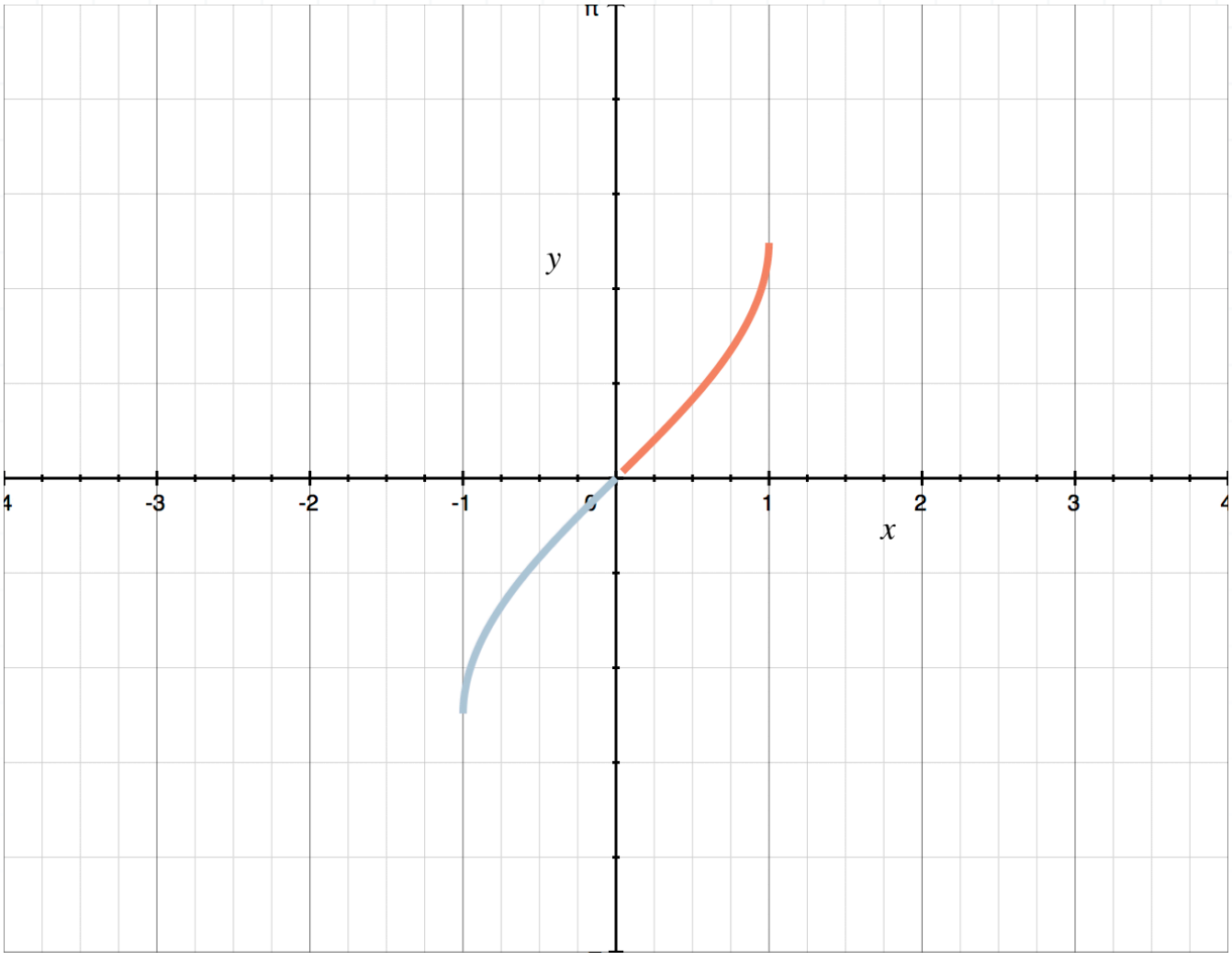
light blue and below the horizontal axis, plus the first half of the part of the graph of the inverse sine relation that's in red and above the horizontal axis.



The graph of the inverse cosine function is identical to the part of the graph of the inverse cosine relation that corresponds to angles in the interval $[0,\pi]$. We'll first display the graph of the inverse cosine relation.

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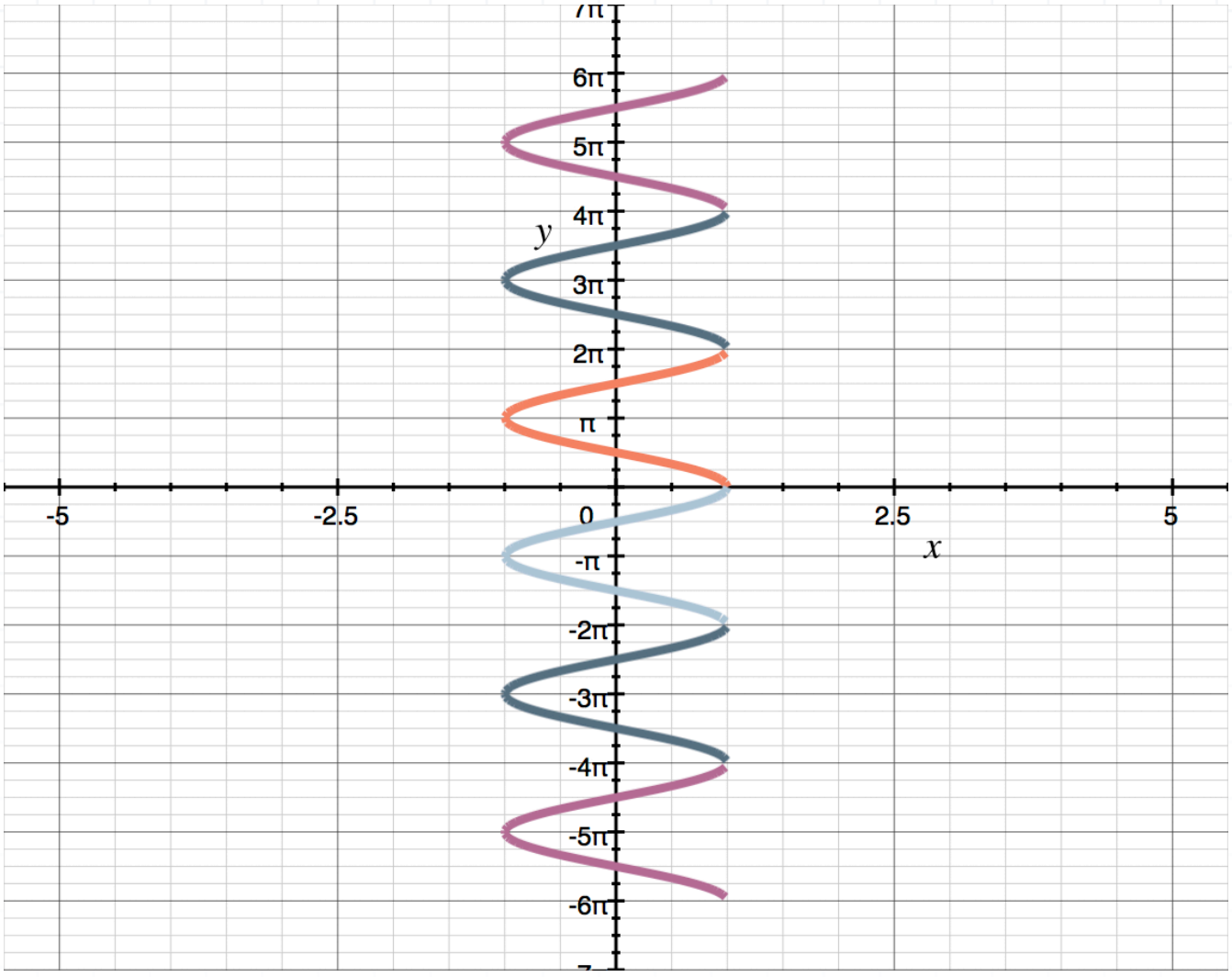

The graph of the inverse cosine function consists of the first half of the part of the graph of the inverse cosine relation that's in red and above the horizontal axis.

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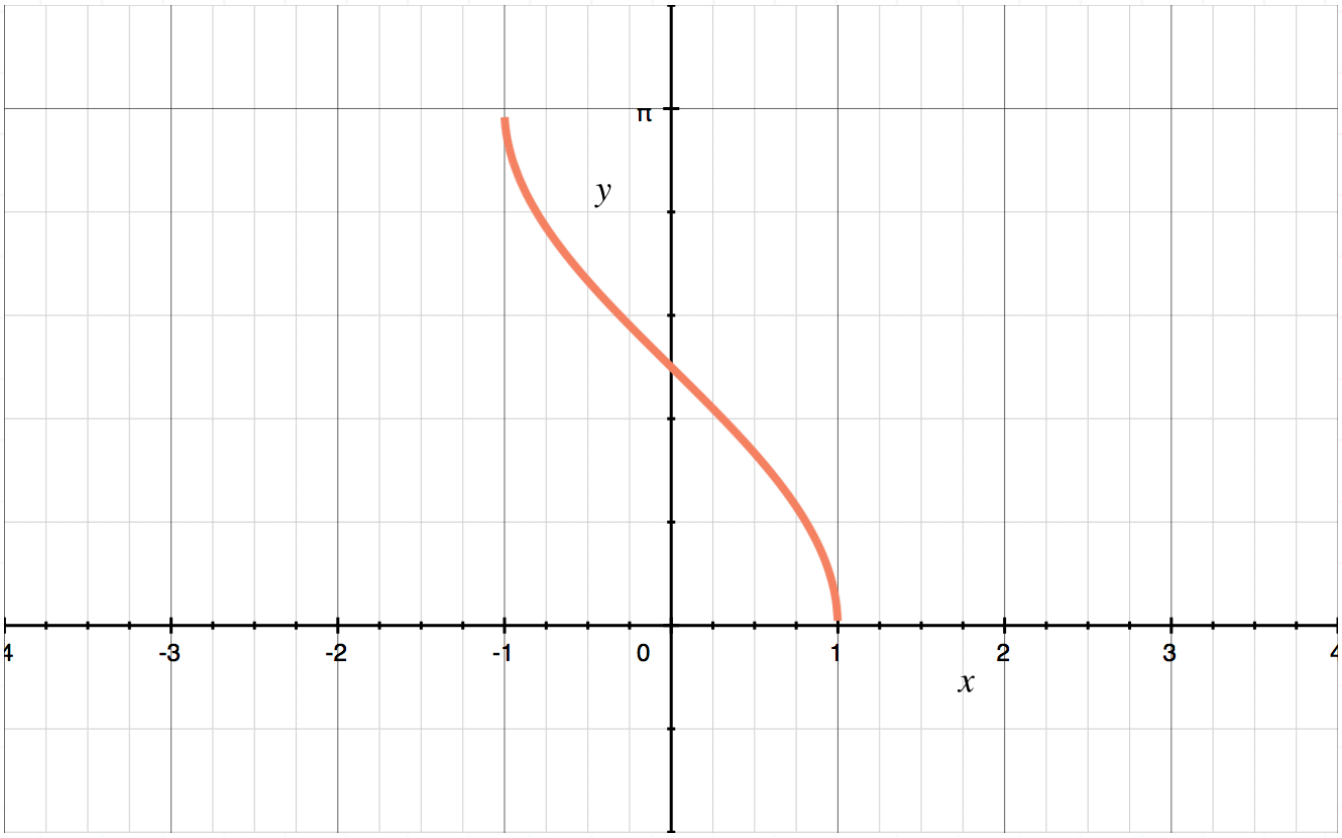

You should get some practice in finding a value of the inverse sine function or the inverse cosine function. To do this, you need to "think backwards" (i.e., to start from a value of the sine function and figure out the angle in the interval $[-\pi/2,\pi/2]$ to which it corresponds, or to start from a value of the cosine function and figure out the angle in the interval $[0,\pi]$ to which it corresponds).

Example

Find the value of $\sin^{-1}\left(-1/\sqrt{2}\right)$.

Recall that

$$\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

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and (by the odd identity for sine)

$$\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

Since $-\pi/4$ is in the interval $[\pi/2,\pi/2]$, you would get

$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$$

Let's do another example, this time with $\cos^{-1}$.

Example

Find the value of $\cos^{-1}(1/2)$.

You may recall that

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

and that $\pi/3$ is in the interval $[0,\pi]$, hence that

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Even if you didn't happen to remember that the angle θ in the interval $[0,\pi]$ whose cosine is equal to $1/2$ is $\pi/3$, you may recall that $\sin(\pi/6) = 1/2$ (or that

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$\sin 30^\circ = 1/2$, and that 30° corresponds to $\pi/6$ radians). The fact that $\sin(\pi/6) = 1/2$, together with the identity

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

would tell you that

$$\frac{1}{2} = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\left(\frac{3(\pi) - 1(\pi)}{6}\right) = \cos\left(\frac{3\pi - \pi}{6}\right) = \cos\left(\frac{2\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$$

Thus

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Actually, you can get along quite easily in trigonometry by memorizing very few facts and identities, perhaps only the following ones:

$$\sin 30^\circ = \frac{1}{2}$$

$180^\circ = \pi$ radians

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

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$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

If you memorize just these few facts and identities, you might be able to derive other results and identities from them (as needed), possibly with the help of the reference angle theorem.

Another trigonometric function that you should be familiar with is the inverse tangent function. As you may recall, the basic tangent function is positive in the first and third quadrants, and it takes on every positive value x in the interval $(0,\infty)$ exactly once in each of those two quadrants. Also, the basic tangent function is negative in the second and fourth quadrants, and it takes on every value x in the interval $(-\infty,0)$ exactly once in each of those two quadrants.

Recall that, by definition of the tangent function,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Thus

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

Also,

$$\cos\left(\frac{\pi}{2}\right) = 0 = \cos\left(-\frac{\pi}{2}\right)$$

so $\tan(\pi/2)$ and $\tan(-\pi/2)$ are undefined.

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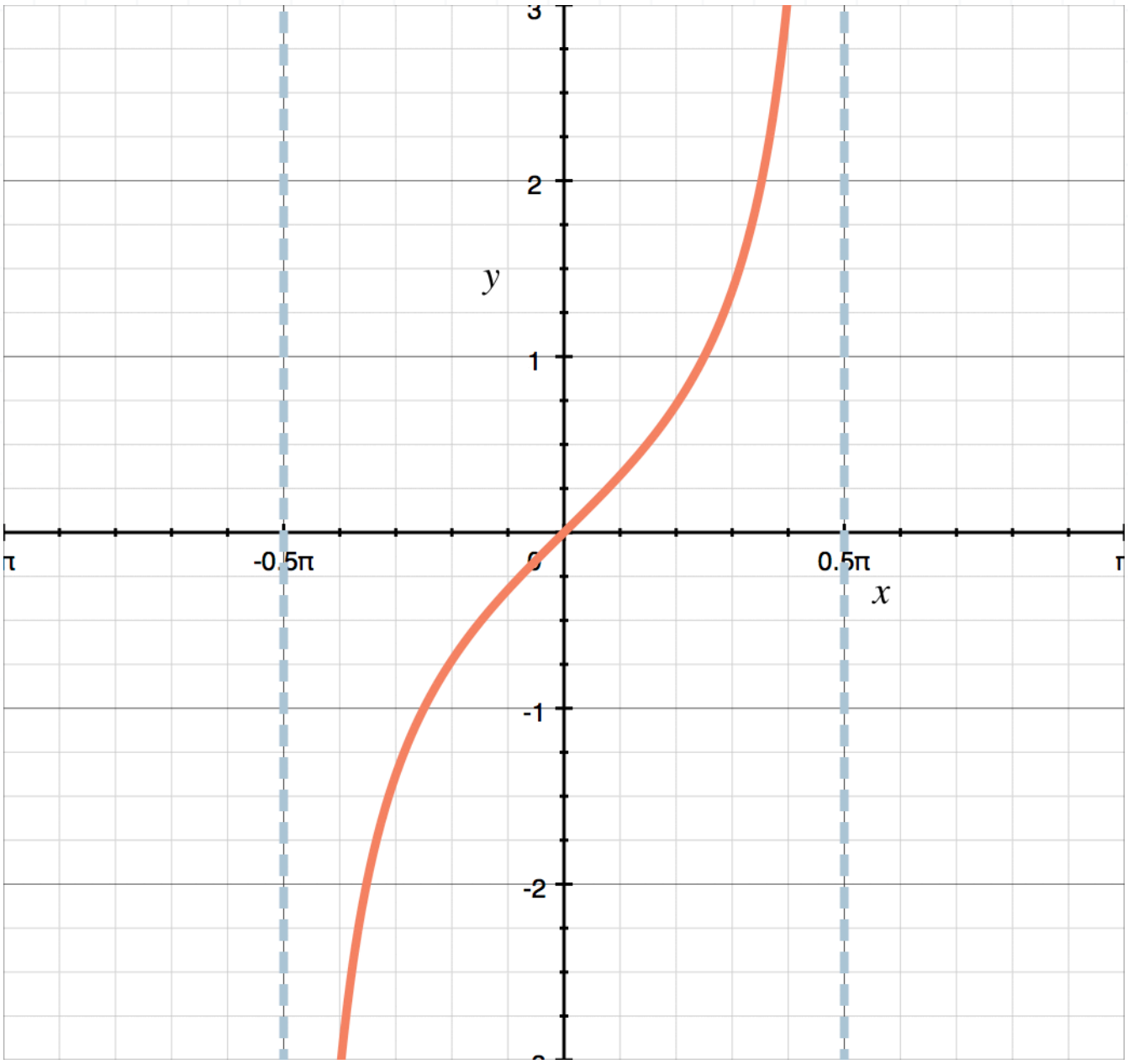
Combining all of these results, we see that the tangent function takes on every value x in the interval $(-\infty, \infty)$, that is, the range of the tangent function is the set of all real numbers. These properties of the tangent function are easily seen from the graph of $\tan\theta$ on the interval $(-\pi/2,\pi/2)$.



Since we'd like the range of the inverse tangent function to be an interval, the natural (and standard) way to restrict the domain of the function $\tan\theta$ (so that the restricted version has an inverse and takes on every real number x exactly once) is to restrict it to the interval $(-\pi/2,\pi/2)$. Thus the domain of the function $\tan^{-1}(x)$ is the set of all real numbers, and its range is the interval $(-\pi/2,\pi/2)$.

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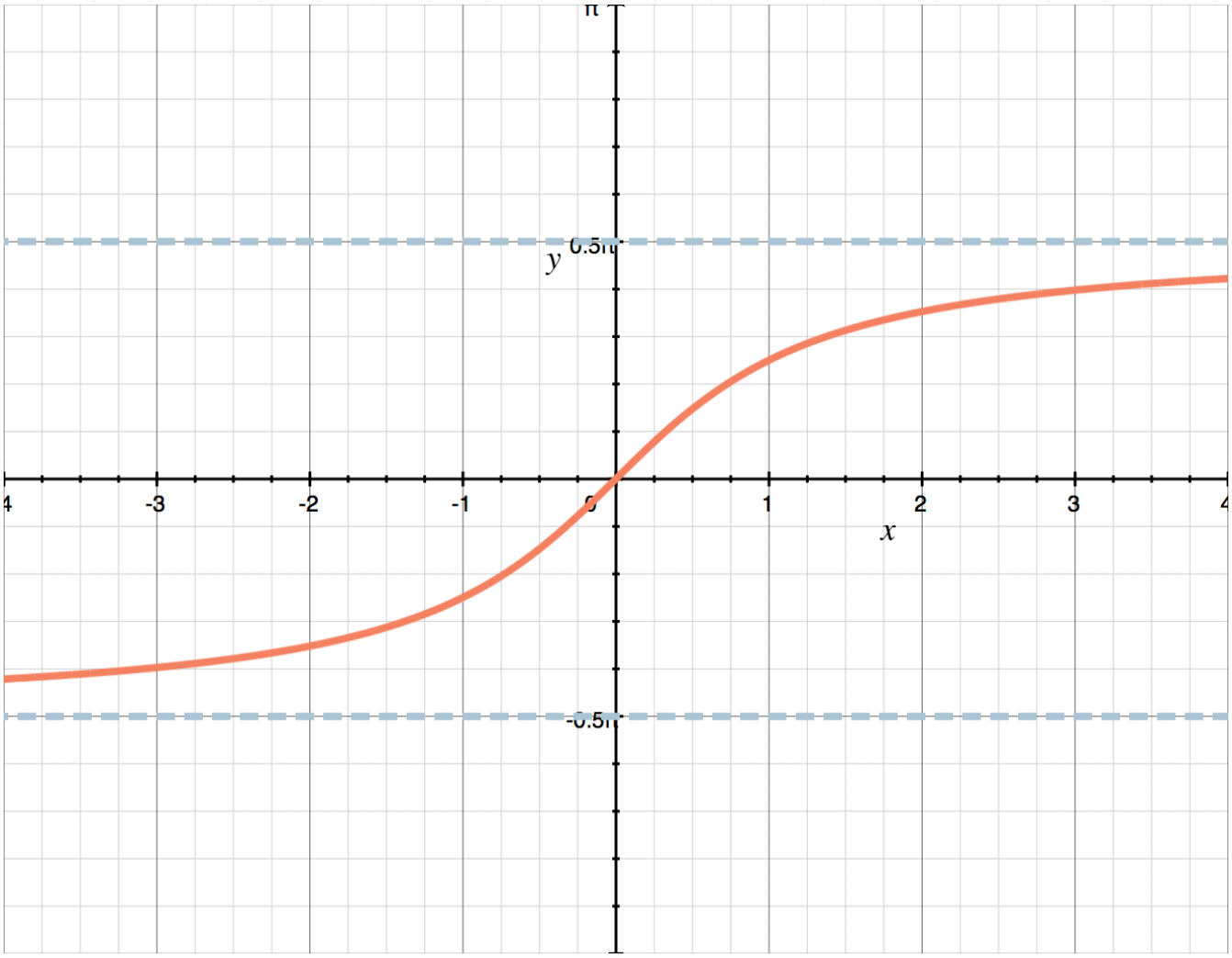
Of course, we can't show the entire graph of the function $\tan^{-1}(x)$, since its domain extends to infinity in both directions (i.e., to the right and to the left), but we can show a portion of it in order to get a general idea of what the entire graph looks like.



The cotangent function also takes on every real number x. You might want to convince yourself that the natural (and standard) way to restrict the domain of the function $\cot\theta$ (so that the restricted version has an inverse and takes on every real number x exactly once) is to restrict it to the interval $(0,\pi)$, hence that the domain of the function $\cot^{-1}(x)$ is the set of all real numbers (and its range is the interval $(0,\pi)$).

Alternative names for the inverse sine function, the inverse cosine function, and the inverse tangent function are the arcsine function, the arccosine function, and the arctangent function, respectively.

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You probably have keys on your calculator that are labeled $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. (Alternatively, they may be labeled $\arcsin$, $\arccos$, and $\arctan$.) When you use those keys, keep in mind that when you input some x in the interval $[-1,1]$, $\sin^{-1}(x)$ (alternatively, $\arcsin(x)$) will compute the angle θ in the interval $[-\pi/2,\pi/2]$ for which $\sin\theta = x$, and $\cos^{-1}(x)$ (alternatively, $\arccos(x)$) will compute the angle θ in the interval $[0,\pi]$ for which $\cos\theta = x$. When you input some real number x, $\tan^{-1}(x)$ (alternatively, $\arctan(x)$) will compute the angle θ in the interval $(-\pi/2,\pi/2)$ for which $\tan\theta = x$.

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